

Eigenfunctions of the Cosine and Sine Transforms

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Abstract

A description of eigensubspaces of the cosine and sine operators is presented. The spectrum of each of these two operator consists of two eigenvalues 1 , -1 and their eigensubspaces are infinite-dimensional. There are many possible bases for these subspaces, but most popular are bases constructed from the Hermite functions. We present other "bases" which are not discrete orthogonal sequences of vectors, but continuous orthogonal chains of vectors. Our work can be considered a continuation and further development of results in *Self-reciprocal functions* by Hardy and Titchmarsh: Quarterly Journ. of Math. (Oxford Ser.) **1** (1930).

Key words: Fourier transform, cosine- sine transforms, eigenfunctions, Melline transform.

Mathematical Subject Classification 2000: Primary 47A38; Secondary 47B35, 47B06, 47A10.

1. The cosine transform \mathfrak{C} and the sine transform \mathfrak{S} are defined by formulas

$$(\mathfrak{C}x)(t) = \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}_+} \cos(t\xi) x(\xi) d\xi, \quad t \in \mathbb{R}_+, \quad (1a)$$

$$(\mathfrak{S}x)(t) = \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}_+} \sin(t\xi) x(\xi) d\xi, \quad t \in \mathbb{R}_+, \quad (1b)$$

where \mathbb{R}_+ is the positive half-axis, $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$.

For $x \in L^1(\mathbb{R}_+)$, the integrals in (1) are well defined as Lebesgue integrals.

If $x(t) \in L^2(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$, then the Parseval equalities hold:

$$\int_{\mathbb{R}_+} |(\mathbf{C}x)(t)|^2 dt = \int_{\mathbb{R}_+} |x(t)|^2 dt, \quad (2a)$$

$$\int_{\mathbb{R}_+} |(\mathbf{S}x)(t)|^2 dt = \int_{\mathbb{R}_+} |x(t)|^2 dt. \quad (2b)$$

Thus, the transforms \mathbf{C} and \mathbf{S} can both be considered as linear operators defined on the linear manifold $L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ of the Hilbert space $L^2(\mathbb{R}_+)$, mapping this linear manifold into $L^2(\mathbb{R}_+)$ *isometrically*. Since the set $L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ is dense in $L^2(\mathbb{R}_+)$, each of these operators can be extended to an operator defined on the *whole* space $L^2(\mathbb{R}_+)$, which maps $L^2(\mathbb{R}_+)$ into $L^2(\mathbb{R}_+)$ isometrically. We retain the notation \mathbf{C} and \mathbf{S} for the extended operators. In an even broader context, the transformation (1) can be considered for those x , for which the the integrals on the right-hand sides are meaningful.

Considered as operators in the Hilbert space $L^2(\mathbb{R}_+)$, the operators \mathbf{C} and \mathbf{S} are self-adjoint operators which satisfy the equalities

$$\mathbf{C}^2 = \mathbf{J}, \quad \mathbf{S}^2 = \mathbf{J}, \quad (3)$$

where \mathbf{J} is the identity operator in $L^2(\mathbb{R}_+)$. Each of the spectra $\sigma(\mathbf{C})$ and $\sigma(\mathbf{S})$ of these operators consist of two points: $+1$ and -1 . By \mathcal{C}_λ and \mathcal{S}_λ we denote the spectral subspaces of the operators \mathbf{C} and \mathbf{S} , respectively, corresponding to the points $\lambda = 1$ and $\lambda = -1$ of their spectra. These spectral subspaces are eigensubspaces:

$$\mathcal{C}_1 = \{x \in L^2(\mathbb{R}_+) : \mathbf{C}x = x\}, \quad \mathcal{C}_{-1} = \{x \in L^2(\mathbb{R}_+) : \mathbf{C}x = -x\}; \quad (4a)$$

$$\mathcal{S}_1 = \{x \in L^2(\mathbb{R}_+) : \mathbf{S}x = x\}, \quad \mathcal{S}_{-1} = \{x \in L^2(\mathbb{R}_+) : \mathbf{S}x = -x\}. \quad (4b)$$

Moreover, two orthogonal decompositions hold:

$$L^2(\mathbb{R}_+) = \mathcal{C}_1 \oplus \mathcal{C}_{-1}, \quad L^2(\mathbb{R}_+) = \mathcal{S}_1 \oplus \mathcal{S}_{-1}. \quad (5)$$

The spectra of the operators \mathbf{C} and \mathbf{S} are highly degenerated: the eigensubspaces \mathcal{C}_λ and \mathcal{S}_λ are infinite-dimensional. Many bases are possible in these

subspaces. The best known are the bases formed by the Hermite functions $h_k(t)$ restricted onto \mathbb{R}_+ .

The Hermite functions $h_k(t)$ are defined as

$$h_k(t) = e^{\frac{t^2}{2}} \frac{d^k (e^{-t^2})}{dt^k}, \quad t \in \mathbb{R}, \quad k = 0, 1, 2, \dots \quad (6)$$

It is known that the system $\{h_k\}_{k=0,1,2,\dots}$ forms an orthogonal basis in the Hilbert space $L^2(\mathbb{R})$. The properties of the Hermite functions h_k as eigenfunctions of the Fourier transform was established by N. Wiener, [1, Chapter 1]. In [1], N. Wiener developed L^2 -theory of the Fourier transform which was based on these properties of the Hermite functions.

The Hermite functions h_k are originally defined on the whole real axis \mathbb{R} . The restrictions $h_k|_{\mathbb{R}_+}$ of the Hermite functions h_k onto \mathbb{R}_+ are considered as vectors of the Hilbert space $L^2(\mathbb{R}_+)$. Each of two systems $\{h_{2k}|_{\mathbb{R}_+}\}_{k=0,1,2,\dots}$ and $\{h_{2k+1}|_{\mathbb{R}_+}\}_{k=0,1,2,\dots}$ is an orthogonal basis in $L^2(\mathbb{R}_+)$. The systems $\{h_{4l}|_{\mathbb{R}_+}\}_{l=0,1,2,\dots}$, $\{h_{4l+2}|_{\mathbb{R}_+}\}_{l=0,1,2,\dots}$, $\{h_{4l+1}|_{\mathbb{R}_+}\}_{l=0,1,2,\dots}$, and $\{h_{4l+3}|_{\mathbb{R}_+}\}_{l=0,1,2,\dots}$ are orthogonal bases of the eigensubspaces \mathcal{C}_1 , \mathcal{C}_{-1} , \mathcal{S}_1 and \mathcal{S}_{-1} , respectively. We present other "bases" which are not discrete orthogonal sequences of vectors, but continuous orthogonal chains of (generalized) vectors. This is the main goal of this paper. Our work may be considered as a further development of results in [2] by Hardy and Titchmarsh. (The contents of [2] and [3] were reproduced in the book [4].)

2. First we discuss eigenfunctions of the transforms \mathfrak{C} and \mathfrak{S} in the broad sense. These transforms are of the form $x \rightarrow \mathfrak{K}x$, where

$$(\mathfrak{K}x)(t) = \int_{\mathbb{R}_+} k(t\xi)x(\xi)d\xi, \quad (7)$$

and k is a function of *one* variable defined on \mathbb{R}_+ . (It should be mentioned that some operational calculus related to operators of the form (7) was developed in [5].)

Remark 1. *If the integral (7) does not exist as a Lebesgue integral, i.e. the function $k(t\xi)x(\xi)$ of the variable ξ is not summable, then a meaning may be attached to the integral (7) by means of some regularization procedure. We use the regularization procedure*

$$\int_{\mathbb{R}_+} k(t\xi)x(\xi)d\xi = \lim_{\varepsilon \rightarrow +0} \int_{\mathbb{R}_+} e^{-\varepsilon\xi} k(t\xi)x(\xi)d\xi, \quad (8)$$

and the regularization procedure

$$\int_{\mathbb{R}_+} k(t\xi)x(\xi)d\xi = \lim_{R \rightarrow +\infty} \int_0^R k(t\xi)x(\xi)d\xi. \quad (9)$$

If for some $a \in \mathbb{C}$ both integrals

$$\int_{\mathbb{R}_+} k(t\xi)\xi^{-a}d\xi \quad \text{and} \quad \int_{\mathbb{R}_+} k(t\xi)\xi^{a-1}d\xi \quad (10)$$

have a meaning for every positive t , then, changing the variable $t\xi \rightarrow \xi$, we obtain

$$\mathfrak{K}t^{-a} = \varkappa(a)t^{a-1}, \quad \mathfrak{K}t^{a-1} = \varkappa(1-a)t^{-a}, \quad (11)$$

where

$$\varkappa(a) = \int_{\mathbb{R}_+} k(\xi)\xi^{-a}d\xi, \quad \varkappa(1-a) = \int_{\mathbb{R}_+} k(\xi)\xi^{a-1}d\xi. \quad (12)$$

The equalities (11) mean that the subspace (two-dimensional if $a \neq 1/2$) generated by the functions t^{-a} and t^{a-1} is invariant with respect to the transformation \mathfrak{K} and that the matrix of this operator in the basis t^{-a}, t^{a-1} is: $\begin{pmatrix} 0 & \varkappa(1-a) \\ \varkappa(a) & 0 \end{pmatrix}$. Thus, assuming that $\varkappa(a) \neq 0$, $\varkappa(1-a) \neq 0$, we obtain that the functions

$$\sqrt{\varkappa(1-a)}t^{-a} + \sqrt{\varkappa(a)}t^{a-1} \quad \text{and} \quad \sqrt{\varkappa(1-a)}t^{-a} - \sqrt{\varkappa(a)}t^{a-1} \quad (13)$$

are the eigenfunctions of the transform \mathfrak{K} corresponding to the eigenvalues

$$\lambda_+ = \sqrt{\varkappa(a)\varkappa(1-a)} \quad \text{and} \quad \lambda_- = -\sqrt{\varkappa(a)\varkappa(1-a)}, \quad (14)$$

respectively.

To find eigenfunctions of the form (13) for cosine and sine transforms \mathfrak{C} and \mathfrak{S} , we have to calculate the constants (12) corresponding to the functions

$$k_c(\tau) = \sqrt{\frac{2}{\pi}} \cos \tau \quad \text{and} \quad k_s(\tau) = \sqrt{\frac{2}{\pi}} \sin \tau, \quad (15)$$

which generate the kernels of these integral transforms. This is accomplished in the following

Lemma 1. *Let ζ belong to the strip $0 < \operatorname{Re} \zeta < 1$.*

Then

$$1. \quad \int_0^\infty (\cos s) s^{\zeta-1} ds = \left(\cos \frac{\pi}{2} \zeta \right) \Gamma(\zeta), \quad (16a)$$

$$\int_0^\infty (\sin s) s^{\zeta-1} ds = \left(\sin \frac{\pi}{2} \zeta \right) \Gamma(\zeta), \quad (16b)$$

where Γ is the Euler Gamma-function and the integrals in (16) are understood in the sense

$$\begin{aligned} & \int_0^\infty \left\{ \begin{matrix} \cos s \\ \sin s \end{matrix} \right\} s^{\zeta-1} ds \\ &= \lim_{R \rightarrow +\infty} \int_0^R \left\{ \begin{matrix} \cos s \\ \sin s \end{matrix} \right\} s^{\zeta-1} ds = \lim_{\varepsilon \rightarrow +0} \int_0^{+\infty} e^{-\varepsilon s} \left\{ \begin{matrix} \cos s \\ \sin s \end{matrix} \right\} s^{\zeta-1} ds; \end{aligned}$$

2. *The above limits exist uniformly with respect to ζ from any fixed compact subset of the strip $0 < \operatorname{Re} \zeta < 1$.*
3. *Given $\delta > 0$, then for any ζ from the strip $\delta < \operatorname{Re} \zeta < 1 - \delta$ and for any $R \in (0, +\infty)$, $\varepsilon \in (0, +\infty)$, the estimates*

$$\left| \int_0^R \left\{ \begin{matrix} \cos s \\ \sin s \end{matrix} \right\} s^{\zeta-1} ds \right| \leq C(\delta) e^{\frac{\pi}{2} |\operatorname{Im} \zeta|}, \quad (17a)$$

$$\left| \int_0^{+\infty} e^{-\varepsilon s} \left\{ \begin{matrix} \cos s \\ \sin s \end{matrix} \right\} s^{\zeta-1} ds \right| \leq C(\delta) e^{\frac{\pi}{2} |\operatorname{Im} \zeta|}, \quad (17b)$$

hold, where $C(\delta) < \infty$ does not depend on ζ , R , and ε .

We omit proof of Lemma 1. This lemma can be proved by a standard method using integration in the complex plane. \square

According to Lemma 1, the integrals

$$\int_0^\infty \left\{ \begin{matrix} k_c(s) \\ k_s(s) \end{matrix} \right\} s^{-a} ds \quad \text{and} \quad \int_0^\infty \left\{ \begin{matrix} k_c(s) \\ k_s(s) \end{matrix} \right\} s^{a-1} ds,$$

where k_c and k_s , (15), are the functions generating the kernels of the integral transformations \mathfrak{C} and \mathfrak{S} , exist for every a such that $0 < \operatorname{Re} a < 1$, or, amounting to the same, $0 < \operatorname{Re}(1 - a) < 1$.

The constants $\varkappa_c(a)$ and $\varkappa_c(1 - a)$, corresponding to the function $k_c(\tau) = \sqrt{\frac{2}{\pi}} \cos \tau$, are:

$$\varkappa_c(a) = \sqrt{\frac{2}{\pi}} \sin \frac{\pi a}{2} \Gamma(1 - a), \quad \varkappa_c(1 - a) = \sqrt{\frac{2}{\pi}} \cos \frac{\pi a}{2} \Gamma(a). \quad (18a)$$

The constants $\varkappa_s(a)$ and $\varkappa_s(1 - a)$, corresponding to the function $k_s(\tau) = \sqrt{\frac{2}{\pi}} \sin \tau$, are:

$$\varkappa_s(a) = \sqrt{\frac{2}{\pi}} \cos \frac{\pi a}{2} \Gamma(1 - a), \quad \varkappa_s(1 - a) = \sqrt{\frac{2}{\pi}} \sin \frac{\pi a}{2} \Gamma(a). \quad (18b)$$

3. Later, we will have to transform the expression (18) for the constants \varkappa_c and \varkappa_s using the following

Identities for the Euler Gamma-function $\Gamma(\zeta)$:

$$\Gamma(\zeta + 1) = \zeta \Gamma(\zeta), \quad \text{see [6], } \mathbf{12.12}, \quad (19a)$$

$$\Gamma(\zeta) \Gamma(1 - \zeta) = \frac{\pi}{\sin \pi \zeta}, \quad \text{see [6], } \mathbf{12.14}, \quad (19b)$$

$$\Gamma(\zeta) \Gamma\left(\zeta + \frac{1}{2}\right) = 2\sqrt{\pi} 2^{-2\zeta} \Gamma(2\zeta), \quad \text{see [6], } \mathbf{12.15}. \quad (19c)$$

Lemma 2. *The following identities hold:*

$$\sqrt{\frac{2}{\pi}} \left(\cos \frac{\pi}{2} \zeta \right) \Gamma(\zeta) = 2^{\zeta - \frac{1}{2}} \frac{\Gamma(\frac{\zeta}{2})}{\Gamma(\frac{1}{2} - \frac{\zeta}{2})}, \quad (20a)$$

$$\sqrt{\frac{2}{\pi}} \left(\sin \frac{\pi}{2} \zeta \right) \Gamma(\zeta) = 2^{\zeta - \frac{1}{2}} \frac{\Gamma(\frac{1}{2} + \frac{\zeta}{2})}{\Gamma(1 - \frac{\zeta}{2})}. \quad (20b)$$

Proof. From (19b) it follows that

$$\cos \frac{\pi}{2} \zeta = \frac{\pi}{\Gamma(\frac{1}{2} - \frac{\zeta}{2}) \Gamma(\frac{1}{2} + \frac{\zeta}{2})}.$$

From (19c) it follows that

$$\Gamma(\zeta) = \pi^{-\frac{1}{2}} \Gamma(\frac{\zeta}{2}) \Gamma(\frac{1}{2} + \frac{\zeta}{2}) 2^{\zeta-1}.$$

Combining the last two formulas, we obtain (20a). Combining the last formula with the formula

$$\sin \frac{\pi}{2} \zeta = \frac{\pi}{\Gamma(\frac{\zeta}{2}) \Gamma(1 - \frac{\zeta}{2})},$$

we obtain (20b). □

Lemma 3. *The values $\varkappa_c(a)$, $\varkappa_c(1-a)$, $\varkappa_s(a)$, $\varkappa_s(1-a)$, which appear as coefficients of linear combinations (13), are*

$$\varkappa_c(a) = 2^{\frac{1}{2}-a} \frac{\Gamma(\frac{1}{2} - \frac{a}{2})}{\Gamma(\frac{a}{2})}, \quad \varkappa_c(1-a) = 2^{a-\frac{1}{2}} \frac{\Gamma(\frac{a}{2})}{\Gamma(\frac{1}{2} - \frac{a}{2})} \quad (21a)$$

$$\varkappa_s(a) = 2^{\frac{1}{2}-a} \frac{\Gamma(1 - \frac{a}{2})}{\Gamma(\frac{1}{2} + \frac{a}{2})}, \quad \varkappa_s(1-a) = 2^{a-\frac{1}{2}} \frac{\Gamma(\frac{1}{2} + \frac{a}{2})}{\Gamma(1 - \frac{a}{2})}. \quad (21b)$$

4. From the expressions (21) we see that the products $\varkappa_c(a)\varkappa_c(1-a)$ and $\varkappa_s(a)\varkappa_s(1-a)$ do not depend on a :

$$\varkappa_c(a)\varkappa_c(1-a) = 1, \quad \varkappa_s(a)\varkappa_s(1-a) = 1 \quad 0 < \operatorname{Re} a < 1.$$

Theorem 1. *Let $a \in \mathbb{C}$, $0 < \operatorname{Re} a < 1$, $a \neq \frac{1}{2}$, and $\varkappa_c(a)$, $\varkappa_c(1-a)$, $\varkappa_s(a)$, $\varkappa_s(1-a)$ be the values which appear in (21).*

Then:

1. *The functions*

$$E_c^+(t, a) = \sqrt{\varkappa_c(1-a)} t^{-a} + \sqrt{\varkappa_c(a)} t^{a-1}, \quad (22a)$$

$$E_c^-(t, a) = \sqrt{\varkappa_c(1-a)} t^{-a} - \sqrt{\varkappa_c(a)} t^{a-1}, \quad (22b)$$

of variable $t \in \mathbb{R}_+$ are eigenfunctions (in a broad sense) of the cosine transform \mathfrak{C} corresponding to the eigenvalues $+1$ and -1 respectively:

$$E_c^+(t, a) = \lim_{R \rightarrow \infty} \sqrt{\frac{2}{\pi}} \int_0^R \cos(t\xi) E_c^+(\xi, a) d\xi,$$

$$E_c^-(t, a) = - \lim_{R \rightarrow \infty} \sqrt{\frac{2}{\pi}} \int_0^R \cos(t\xi) E_c^-(\xi, a) d\xi;$$

2. The functions

$$E_s^+(t, a) = \sqrt{\varkappa_s(1-a)} t^{-a} + \sqrt{\varkappa_s(a)} t^{a-1}, \quad (23a)$$

$$E_s^-(t, a) = \sqrt{\varkappa_s(1-a)} t^{-a} - \sqrt{\varkappa_s(a)} t^{a-1} \quad (23b)$$

of variable $t \in \mathbb{R}_+$ are eigenfunctions (in a broad sense) of the sine transform \mathfrak{S} corresponding to the eigenvalues $+1$ and -1 respectively:

$$E_s^+(t, a) = \lim_{R \rightarrow \infty} \sqrt{\frac{2}{\pi}} \int_0^R \sin(t\xi) E_s^+(\xi, a) d\xi, \quad (24)$$

$$E_s^-(t, a) = - \lim_{R \rightarrow \infty} \sqrt{\frac{2}{\pi}} \int_0^R \sin(t\xi) E_s^-(\xi, a) d\xi. \quad (25)$$

For fixed $t \in (0, \infty)$, the limits exist uniformly with respect to a , from any compact subset of the strip $0 < \operatorname{Re} a < 1$.

Remark 2. In (22) and (23), the values of the square roots $\sqrt{\varkappa(a)}$ and $\sqrt{\varkappa(1-a)}$ should be chosen such that their products equal 1.

Remark 3. For $a = \frac{1}{2}$ there is only one eigenfunction

$$E(t, \tfrac{1}{2}) = 2t^{-\frac{1}{2}}.$$

Remark 4. Since

$$E_c^+(t, a) = E_c^+(t, 1-a), \quad E_c^-(t, a) = -E_c^-(t, 1-a), \quad (26a)$$

$$E_s^+(t, a) = E_s^+(t, 1-a), \quad E_s^-(t, a) = -E_s^-(t, 1-a), \quad (26b)$$

each eigenfunction appears in the family $\{E_{c,s}^\pm(t, a)\}_{0 < \operatorname{Re} a < 1}$ twice. To avoid this redundancy, we should consider the family where only one of the points a or $1-a$ appear.

5. If $0 < \operatorname{Re} a < 1$ and $x(t)$ is any of the eigenfunctions of the form either (22), or (23), then the integral $\int_{\mathbb{R}_+} |x(t)|^2$ diverges. Thus, none of these eigenfunctions belong to $L^2(\mathbb{R}_+)$. This integral diverges both at points $t = +0$ and at point $t = +\infty$. However, this integral diverges variously for a with $\operatorname{Re} a = \frac{1}{2}$ and for a with $\operatorname{Re} a \neq \frac{1}{2}$. If $\operatorname{Re} a = \frac{1}{2}$, then the integrals diverge *logarithmically* both at $t = +0$ and at $t = +\infty$. If $\operatorname{Re} a \neq \frac{1}{2}$, then the integrals diverge more strongly: *powerwise*. We try to construct eigenfunctions of the operator \mathfrak{C} (of operator \mathfrak{S}) from L^2 as continuous combinations of the eigenfunctions of the form (22) (of the form (23)). Our hope is that singularities of "continuous linear combinations" of eigenfunctions, which are in some sense an *averaging* of eigenfunctions of the family, are weaker than singularities of individual eigenfunctions. Such continuous linear combinations should *not* include eigenfunctions of the form (22) and (23) with a : $\operatorname{Re} a \neq \frac{1}{2}$. Singularities of eigenfunctions with a : $\operatorname{Re} a \neq \frac{1}{2}$ are too strong and can not disappear by averaging. Thus, we have to restrict ourselves to a 's of the form $a = \frac{1}{2} + i\tau$, $\tau \in \mathbb{R}$.

Considering the case $\operatorname{Re} a = \frac{1}{2}$ in more detail, we introduce special notation for the eigenfunctions $E_{c,s}^\pm(t, \frac{1}{2} + i\tau)$:

$$e_c^+(t, \tau) = \frac{1}{2\sqrt{\pi}} E_c^+(t, \frac{1}{2} + i\tau), \quad e_c^-(t, \tau) = \frac{1}{2i\sqrt{\pi}} E_c^-(t, \frac{1}{2} + i\tau), \quad (27a)$$

$$e_s^+(t, \tau) = \frac{1}{2\sqrt{\pi}} E_s^+(t, \frac{1}{2} + i\tau), \quad e_s^-(t, \tau) = \frac{1}{2i\sqrt{\pi}} E_s^-(t, \frac{1}{2} + i\tau). \quad (27b)$$

(We include the normalizing factor $\frac{1}{2\sqrt{\pi}}$ in the definition of the functions $e_{c,s}^\pm$.) According to (21), (22), the functions $e_{c,s}^\pm(t, \tau)$ can be expressed as

$$e_c^+(t, \tau) = \frac{1}{2\sqrt{\pi}} \left(t^{-\frac{1}{2}-i\tau} c(\tau) + t^{-\frac{1}{2}+i\tau} c(-\tau) \right), \quad (28a)$$

$$e_c^-(t, \tau) = \frac{1}{2i\sqrt{\pi}} \left(t^{-\frac{1}{2}-i\tau} c(\tau) - t^{-\frac{1}{2}+i\tau} c(-\tau) \right), \quad (28b)$$

$$e_s^+(t, \tau) = \frac{1}{2\sqrt{\pi}} \left(t^{-\frac{1}{2}-i\tau} s(\tau) + t^{-\frac{1}{2}+i\tau} s(-\tau) \right), \quad (29a)$$

$$e_s^-(t, \tau) = \frac{1}{2i\sqrt{\pi}} \left(t^{-\frac{1}{2}-i\tau} s(\tau) - t^{-\frac{1}{2}+i\tau} s(-\tau) \right), \quad (29b)$$

where $c(\tau)$, $s(\tau)$ are "phase factors":

$$c(\tau) = 2^{i\frac{\tau}{2}} \exp \left\{ i \arg \Gamma\left(\frac{1}{4} + i\frac{\tau}{2}\right) \right\}, \quad -\infty < \tau < \infty, \quad (30a)$$

$$s(\tau) = 2^{i\frac{\tau}{2}} \exp \left\{ i \arg \Gamma \left(\frac{3}{4} + i\frac{\tau}{2} \right) \right\}, \quad -\infty < \tau < \infty. \quad (30b)$$

In (30), $\exp \{ i \arg \Gamma(\zeta) \} = \frac{\Gamma(\zeta)}{|\Gamma(\zeta)|}$.

Since $c(\tau) = \overline{c(-\tau)}$, $s(\tau) = \overline{s(-\tau)}$ for real τ , the values of the functions $e_c^+(t, \tau)$, $e_c^-(t, \tau)$, $e_s^+(t, \tau)$, $e_s^-(t, \tau)$ are *real* for $t \in (0, \infty)$, $\tau \in (0, \infty)$.

Remark 5. *The parameter τ , which enumerates the families $\{e_c^\pm(t, \tau)\}$, $\{e_s^\pm(t, \tau)\}$, runs over the interval $(0, \infty)$. There is no need to consider negative τ . (See Remark 4).*

6. Let us introduce four integral transforms \mathfrak{T}_c^+ , \mathfrak{T}_c^- , \mathfrak{T}_s^+ , \mathfrak{T}_s^- . For $\phi(t) \in L^1(\mathbb{R}_+)$ and $t > 0$, let us define

$$(\mathfrak{T}_c^+ \phi)(t) = \int_{\mathbb{R}_+} e_c^+(t, \tau) \phi(\tau) d\tau, \quad (\mathfrak{T}_c^- \phi)(t) = \int_{\mathbb{R}_+} e_c^-(t, \tau) \phi(\tau) d\tau, \quad (31a)$$

$$(\mathfrak{T}_s^+ \phi)(t) = \int_{\mathbb{R}_+} e_s^+(t, \tau) \phi(\tau) d\tau, \quad (\mathfrak{T}_s^- \phi)(t) = \int_{\mathbb{R}_+} e_s^-(t, \tau) \phi(\tau) d\tau, \quad (31b)$$

Lemma 4. *If $\phi(\tau) \in L^1(\mathbb{R}_+)$, and $x(t) = (\mathfrak{T}\phi)(t)$, where \mathfrak{T} is any of the above-introduced four transformations $\mathfrak{T}_{c,s}^\pm$, then the function $x(t)$ is continuous on the interval $(0, \infty)$ and the estimate*

$$|x(t)| \leq \frac{1}{\sqrt{\pi}} \|\phi\|_{L^1(\mathbb{R}_+)} \cdot t^{-\frac{1}{2}}, \quad 0 < t < \infty, \quad (32)$$

holds.

Proof. Let $e(t, \tau)$ be any of the four above-introduced functions $e_c^+(t, \tau)$, $e_c^-(t, \tau)$, $e_s^+(t, \tau)$, $e_s^-(t, \tau)$. The function $e(t, \tau)$ is continuous with respect to t at each $t > 0$, $\tau > 0$ and satisfies the estimate

$$|e(t, \tau)| \leq \frac{1}{\sqrt{\pi}} t^{-\frac{1}{2}}, \quad 0 < t < \infty, \quad 0 < \tau < \infty. \quad (33)$$

Now Lemma 4 is a consequence of standard results of Lebesgue integration theory. \square

Theorem 2. *Let $\phi(\tau)$ be a function, satisfying the condition*

$$\int_0^\infty |\phi(\tau)| e^{\frac{\pi}{2}\tau} d\tau < \infty. \quad (34)$$

and

$$x_c^+(t) = (\mathfrak{T}_c^+ \phi)(t), \quad x_c^-(t) = (\mathfrak{T}_c^- \phi)(t), \quad (35)$$

$$x_s^+(t) = (\mathfrak{T}_s^+ \phi)(t), \quad x_s^-(t) = (\mathfrak{T}_s^- \phi)(t), \quad (36)$$

Then the functions $x_c^+(t)$, $x_c^-(t)$ are eigenfunctions (in the broad sense) of the cosine transform \mathfrak{C} and the functions $x_s^+(t)$, $x_s^-(t)$ are eigenfunctions (in the broad sense) of the sine transform \mathfrak{S} , i.e.

$$x_c^+(t) = \lim_{R \rightarrow \infty} \sqrt{\frac{2}{\pi}} \int_0^R \cos(t\xi) x_c^+(\xi) d\xi, \quad (37a)$$

$$x_c^-(t) = - \lim_{R \rightarrow \infty} \sqrt{\frac{2}{\pi}} \int_0^R \cos(t\xi) x_c^-(\xi) d\xi. \quad (37b)$$

and

$$x_s^+(t) = \lim_{R \rightarrow \infty} \sqrt{\frac{2}{\pi}} \int_0^R \sin(t\xi) x_s^+(\xi) d\xi, \quad (38a)$$

$$x_s^-(t) = - \lim_{R \rightarrow \infty} \sqrt{\frac{2}{\pi}} \int_0^R \sin(t\xi) x_s^-(\xi) d\xi. \quad (38b)$$

for every $t \in (0, \infty)$. In particular, in (37), (38) the limits exist.

Proof. According to Theorem 1 and (27),

$$e_c^+(t, \tau) = \lim_{R \rightarrow \infty} \sqrt{\frac{2}{\pi}} \int_0^R \cos(t\xi) e_c^+(\xi, \tau) d\xi \quad \text{for every } t, \tau.$$

Multiplying by $\phi(\tau)$ and integrating with respect to τ , we obtain

$$x_c^+(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\lim_{R \rightarrow \infty} \int_0^R \cos(t\xi) e_c^+(\xi, \tau) d\xi \right) \phi(\tau) d\tau.$$

From (17) we obtain the estimate

$$\left| \int_0^R \cos(t\xi) e_c^+(\xi, \tau) d\xi \right| \leq Ct^{-\frac{1}{2}} e^{\frac{\pi}{2}\tau}, \quad \forall R < \infty, \tau \in \mathbb{R}_+, t \in \mathbb{R}_+,$$

where the value $C < \infty$ does not depend on R, τ, t . This estimate and condition (34) for the function $\phi(t)$ allow us to apply the Lebesgue dominated convergence theorem:

$$\begin{aligned} \int_0^\infty \left(\lim_{R \rightarrow \infty} \int_0^R \cos(t\xi) e_c^+(\xi, \tau) d\xi \right) \phi(\tau) d\tau = \\ \lim_{R \rightarrow \infty} \int_0^\infty \left(\int_0^R \cos(t\xi) e_c^+(\xi, \tau) d\xi \right) \phi(\tau) d\tau. \end{aligned} \quad (39)$$

Thus,

$$x_c^+(t) = \lim_{R \rightarrow \infty} \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\int_0^R \cos(t\xi) e_c^+(\xi, \tau) d\xi \right) \phi(\tau) d\tau.$$

On the other hand, using the estimate (33) for $e_c^+(\xi, \tau)$, we can justify the change of order of integration in the series integral which appears on the right-hand side of the above equality. For any finite R ,

$$\begin{aligned} \int_0^\infty \left(\int_0^R \cos(t\xi) e_c^+(\xi, \tau) d\xi \right) \phi(\tau) d\tau = \\ \int_0^R \cos(t\xi) \left(\int_0^\infty e_c^+(\xi, \tau) \phi(\tau) d\tau \right) d\xi = \int_0^R \cos(t\xi) x_c^+(\xi) d\xi. \end{aligned}$$

Finally, we obtain the equality $x_c^+(t) = \lim_{R \rightarrow \infty} \int_0^R \cos(t\xi) x_c^+(\xi) d\xi$, i.e. the equality (37a) for the function x_c^+ . The equality (37b) for the function x_c^- and the equalities (38) for the functions x_s^+, x_s^- can be obtained analogously. \square

Remark 6. In Theorem 2 we assume that the function ϕ satisfies condition (34). Assuming only that $\int_0^\infty |\phi(\tau)| d\tau < \infty$, we can not justify the equality (39). To apply the Lebesgue dominated convergence theorem, we need the estimate

$$\sup_{\substack{R \in (0, \infty) \\ \tau \in (-\infty, \infty)}} \left| \int_0^R (\cos \xi) \cdot \xi^{-\frac{1}{2} + i\tau} d\xi \right| < \infty.$$

We are, however, able to establish (17), but this estimate is not strong enough.

The question of whether the equalities (37), (38) hold under the assumption $\int_0^\infty |\phi(\tau)| d\tau < \infty$ remains open.

7. Our considerations in the context of L^2 -theory on the operators \mathfrak{C} and \mathfrak{S} are based on L^2 -theory for the Melline transform. (See the article "Melline Transform" on page 192 of [7, Volume 6] and references there.) The Melline transform \mathfrak{M} is defined by

$$(\mathfrak{M}f)(\zeta) = \int_0^\infty f(t)t^{\zeta-1} dt.$$

If the function $f(t) \in L^2(\mathbb{R}_+)$ is *compactly* supported in the *open* interval $(0, \infty)$, then the function $\Phi(\zeta) = (\mathfrak{M}f)(\zeta)$ of variable ζ is defined in the whole complex ζ -plane and holomorphic there. The function $f(t)$ can be recovered from the function $\Phi = \mathfrak{M}f$ by the formula

$$f(t) = \frac{1}{2\pi i} \int_{\operatorname{Re} \zeta = c} \Phi(\zeta) t^{-\zeta} d\zeta,$$

where c is an arbitrary real number. Moreover, the Parseval equality

$$\int_0^\infty |f(t)|^2 dt = \frac{1}{2\pi} \int_{\operatorname{Re} \zeta = \frac{1}{2}} |\Phi(\zeta)|^2 |d\zeta|$$

holds (from which we recognize the significance of the vertical line $\operatorname{Re} \zeta = \frac{1}{2}$). Thus the Melline transform \mathfrak{M} generates the linear operator defined on the set of all compactly supported functions f from $L^2(\mathbb{R}_+)$ which maps this set isometrically into the space $L^2(\operatorname{Re} \zeta = \frac{1}{2})$ of functions defined on the vertical line $\operatorname{Re} \zeta = \frac{1}{2}$ and which are square-integrable there. Since the set of all compactly supported functions f is dense in $L^2(\mathbb{R}_+)$, this operator can be extended to an isometrical operator defined on the whole $L^2(\mathbb{R}_+)$ which maps $L^2(\mathbb{R}_+)$ isometrically into $L^2(\operatorname{Re} \zeta = \frac{1}{2})$. We will continue to denote this extended operator by \mathfrak{M} .

It turns out that the operator \mathfrak{M} maps the space $L^2(\mathbb{R}_+)$ *onto* the whole space $L^2(\operatorname{Re} \zeta = \frac{1}{2})$. The inverse operator \mathfrak{M}^{-1} is defined *everywhere* on

$L^2(\operatorname{Re} \zeta = \frac{1}{2})$. If $\Phi \in L^2(\operatorname{Re} \zeta = \frac{1}{2})$, then the function

$$f(t) = (\mathfrak{M}^{-1}\Phi)(t)$$

is defined as an $L^2(\mathbb{R}_+)$ -function and can be expressed as

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi\left(\frac{1}{2} + i\tau\right) t^{-\frac{1}{2} - i\tau} d\tau, \quad 0 < t < \infty. \quad (40a)$$

Furthermore, the function

$$\Phi\left(\frac{1}{2} + i\tau\right) = (\mathfrak{M}f)\left(\frac{1}{2} + i\tau\right)$$

can be expressed as

$$\Phi\left(\frac{1}{2} + i\tau\right) = \int_0^{\infty} f(t) t^{-\frac{1}{2} + i\tau} dt, \quad -\infty < \tau < \infty. \quad (40b)$$

The pair of formulas (40a) and (40b) together with the Parseval equality

$$\int_0^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\Phi\left(\frac{1}{2} + i\tau\right)|^2 d\tau \quad (40c)$$

make up the most important part of the L^2 -theory of Melline transform.

8. Developing L^2 -theory of the cosine and sine transforms, we first of all prove

Lemma 5. *Let $\phi(t) \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$. Then*

$$\int_{\mathbb{R}_+} |(\mathfrak{T}\phi)(t)|^2 dt = \int_{\mathbb{R}_+} |\phi(\tau)|^2 d\tau, \quad (41)$$

where \mathfrak{T} is any of the above-introduced (see (31)) four transformations $\mathfrak{T}_{c,s}^{\pm}$.

Proof. The proof is based on the Parseval equality for the Melline transform. We present the transformations $\mathfrak{T}_{c,s}^{\pm}$ as inverse Melline transforms. Given a function $\phi(\tau)$ defined for $\tau \in (0, \infty)$, we introduce the functions

$$\Phi_c^+\left(\frac{1}{2} + i\tau\right) = \sqrt{\pi} c(\tau) \phi(|\tau|), \quad (42a)$$

$$\Phi_c^-\left(\frac{1}{2} + i\tau\right) = \frac{1}{i} \operatorname{sign}(\tau) \sqrt{\pi} c(\tau) \phi(|\tau|), \quad (42b)$$

and

$$\Phi_s^+\left(\frac{1}{2} + i\tau\right) = \sqrt{\pi} s(\tau) \phi(|\tau|), \quad (43a)$$

$$\Phi_s^-\left(\frac{1}{2} + i\tau\right) = \frac{1}{i} \operatorname{sign}(\tau) \sqrt{\pi} s(\tau) \phi(|\tau|), \quad (43b)$$

which are defined for $\tau \in (-\infty, \infty)$. Here $c(\tau)$, $s(\tau)$ are the "phase factors" introduced in (30). It is clear that $|\Phi(\frac{1}{2} + i\tau)| = \sqrt{\pi} \phi(|\tau|)$, thus

$$\int_{-\infty}^{\infty} |\Phi(\frac{1}{2} + i\tau)|^2 d\tau = 2\pi \int_0^{\infty} |\phi(\tau)|^2 d\tau,$$

where Φ is any of the four functions Φ_c^+ , Φ_c^- , Φ_s^+ , Φ_s^- . Comparing (31a), (28a) and (42a), we see that the function $(\mathcal{T}_c^+ \phi)(t)$ can be interpreted as the inverse Melline transform of the function Φ_c^+ :

$$(\mathcal{T}_c^+ \phi)(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} t^{-\frac{1}{2} - i\tau} \Phi_c^+\left(\frac{1}{2} + i\tau\right) d\tau. \quad (44a)$$

The Parseval equality transform, as applied to the inverse Melline transform of the function $\varphi_c^+(\tau)$, yields:

$$\int_0^{\infty} |(\mathcal{T}_c^+ \phi)(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\Phi_c^+\left(\frac{1}{2} + i\tau\right)|^2 d\tau = \int_0^{\infty} |\phi(\tau)|^2 d\tau.$$

This is equality (41) for the transform \mathcal{T}_c^+ .

The functions $\mathcal{T}_c^- \phi$, $\mathcal{T}_s^+ \phi$, $\mathcal{T}_s^- \phi$ can also be interpreted as inverse Melline transforms:

$$(\mathcal{T}_c^- \phi)(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} t^{-\frac{1}{2} - i\tau} \Phi_c^-\left(\frac{1}{2} + i\tau\right) d\tau, \quad (44b)$$

and

$$(\mathcal{T}_s^+ \phi)(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} t^{-\frac{1}{2} - i\tau} \Phi_s^+\left(\frac{1}{2} + i\tau\right) d\tau, \quad (45a)$$

$$(\mathcal{T}_s^-\phi)(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} t^{-\frac{1}{2}-i\tau} \Phi_s^-\left(\frac{1}{2} + i\tau\right) d\tau. \quad (45b)$$

The Parseval equalities, as applied to the inverse Melline transform of the functions Φ_c^- , Φ_s^+ and Φ_s^- , yield the equalities (41) for the transforms \mathcal{T}_c^- , \mathcal{T}_s^+ and \mathcal{T}_s^- , respectively. \square

9. According to Lemma 5, the operators \mathcal{T}_c^+ , \mathcal{T}_c^- , \mathcal{T}_s^+ , \mathcal{T}_s^- are linear operators each of which is defined on the linear manifold $L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ of the Hilbert space $L^2(\mathbb{R}_+)$ and which maps this linear manifold into $L^2(\mathbb{R}_+)$ *isometrically*. Since the set $L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ is dense in $L^2(\mathbb{R}_+)$, each of these operators can be extended to an operator defined on the whole space $L^2(\mathbb{R}_+)$, which maps $L^2(\mathbb{R}_+)$ into $L^2(\mathbb{R}_+)$ isometrically. We will continue to write \mathcal{T}_c^+ , \mathcal{T}_c^- , \mathcal{T}_s^+ and \mathcal{T}_s^- for the extended operators.

We now consider the operators \mathcal{T}_c^+ , \mathcal{T}_c^- , \mathcal{T}_s^+ , \mathcal{T}_s^- as operators defined on *all* of $L^2(\mathbb{R}_+)$, mapping $L^2(\mathbb{R}_+)$ into $L^2(\mathbb{R}_+)$ isometrically and acting on the functions $\phi(t) \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ according to (31).

Theorem 3.

1. *The range of values of the operator \mathcal{T}_c^+ is the eigensubspace \mathcal{C}_{+1} of the operator \mathcal{C} ;*
2. *The range of values of the operator \mathcal{T}_c^- is the eigensubspace \mathcal{C}_{-1} of the operator \mathcal{C} ;*
3. *The range of values of the operator \mathcal{T}_s^+ is the eigensubspace \mathcal{S}_{+1} of the operator \mathcal{S} ;*
4. *The range of values of the operator \mathcal{T}_s^- is the eigensubspace \mathcal{S}_{-1} of the operator \mathcal{S} .*

Remark 7. *Since the operators \mathcal{T}_c^+ , \mathcal{T}_c^- , \mathcal{T}_s^+ , \mathcal{T}_s^- act isometrically from $L^2(\mathbb{R}_+)$ into $L^2(\mathbb{R}_+)$, the equalities*

$$(\mathcal{T}_c^+)^* \mathcal{T}_c^+ = \mathcal{J}, \quad \mathcal{T}_c^+(\mathcal{T}_c^+)^* = \mathcal{P}_c^+, \quad \mathcal{C}\mathcal{T}_c^+ = \mathcal{T}_c^+; \quad (46a)$$

$$(\mathcal{T}_c^-)^* \mathcal{T}_c^- = \mathcal{J}, \quad \mathcal{T}_c^-(\mathcal{T}_c^-)^* = \mathcal{P}_c^-, \quad \mathcal{C}\mathcal{T}_c^- = -\mathcal{T}_c^-. \quad (46b)$$

and

$$(\mathcal{T}_s^+)^* \mathcal{T}_s^+ = \mathcal{J}, \quad \mathcal{T}_s^+(\mathcal{T}_s^+)^* = \mathcal{P}_s^+, \quad \mathcal{S}\mathcal{T}_s^+ = \mathcal{T}_s^+; \quad (47a)$$

$$(\mathcal{T}_s^-)^* \mathcal{T}_s^- = \mathcal{I}, \quad \mathcal{T}_s^- (\mathcal{T}_s^-)^* = \mathcal{P}_s^-, \quad \mathcal{S} \mathcal{T}_s^- = -\mathcal{T}_s^-. \quad (47b)$$

hold, where \mathcal{P}_c^+ , \mathcal{P}_c^- , \mathcal{P}_s^+ and \mathcal{P}_s^- are orthogonal projectors from $L^2(\mathbb{R})_+$ onto the eigensubspaces \mathcal{C}_{+1} , \mathcal{C}_{-1} , \mathcal{S}_{+1} and \mathcal{S}_{-1} , respectively and $(\mathcal{T}_c^+)^*$, $(\mathcal{T}_c^-)^*$, $(\mathcal{T}_s^+)^*$, $(\mathcal{T}_s^-)^*$ are the operators Hermitian-conjugated to the operators (\mathcal{T}_c^+) , (\mathcal{T}_c^-) , (\mathcal{T}_s^+) , (\mathcal{T}_s^-) with respect to the standard scalar product in the Hilbert space $L^2(\mathbb{R}_+)$.

In particular, the operators $(\mathcal{T}_c^+)^*$, $(\mathcal{T}_c^-)^*$, $(\mathcal{T}_s^+)^*$ and $(\mathcal{T}_s^-)^*$ are generalized inverses¹ of the operators \mathcal{T}_c^+ , \mathcal{T}_c^- , \mathcal{T}_s^+ and \mathcal{T}_s^- , respectively.

It with mentioning that

$$((\mathcal{T}_c^+)^* x)(\tau) = \int_{\mathbb{R}_+} e_c^+(t, \tau) x(t) dt, \quad ((\mathcal{T}_c^-)^* x)(\tau) = \int_{\mathbb{R}_+} e_c^-(t, \tau) x(t) dt, \quad (48a)$$

$$((\mathcal{T}_s^+)^* x)(\tau) = \int_{\mathbb{R}_+} e_s^+(t, \tau) x(t) dt, \quad ((\mathcal{T}_s^-)^* x)(\tau) = \int_{\mathbb{R}_+} e_s^-(t, \tau) x(t) dt. \quad (48b)$$

Theorem 3 is a consequence of the following

Lemma 6. *Let a function $x(t)$ belong to $L^2(\mathbb{R}_+)$ and $\hat{x}_c(t)$ and $\hat{x}_s(t)$ be the cosine and sine Fourier transform of the function x :*

$$\hat{x}_c(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty x(s) \cos(ts) ds, \quad (49a)$$

$$\hat{x}_s(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty x(s) \sin(ts) ds \quad (49b)$$

Let $\Phi_x(\zeta)$, $\Phi_{\hat{x}_c}(\zeta)$ and $\Phi_{\hat{x}_s}(\zeta)$ be the Melline transforms of the functions x , \hat{x}_c and \hat{x}_s , respectively. (All three functions x , \hat{x}_c , \hat{x}_s belong to $L^2(0, \infty)$, so their Melline transforms exist and are L^2 functions on the vertical line $\text{Re } \zeta = \frac{1}{2}$.)

Then for $\zeta : \text{Re } \zeta = \frac{1}{2}$, the equalities

$$\Phi_{\hat{x}_c}(\zeta) = \Phi_x(1 - \zeta) \cdot 2^{\zeta - \frac{1}{2}} \frac{\Gamma(\frac{\zeta}{2})}{\Gamma(\frac{1}{2} - \frac{\zeta}{2})}, \quad (50a)$$

¹ In the sense of Moore-Penrose, for example.

$$\Phi_{\hat{x}_s}(\zeta) = \Phi_x(1 - \zeta) \cdot 2^{\zeta - \frac{1}{2}} \frac{\Gamma(\frac{1}{2} + \frac{\zeta}{2})}{\Gamma(1 - \frac{\zeta}{2})}. \quad (50b)$$

hold.

Proof. It is enough to prove the equalities (50) assuming that the functions $x(t), \hat{x}_c(t), \hat{x}_s(t)$ are continuous and belong to $L^2(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$: the set of such functions x is dense in $L^2(\mathbb{R})$ and all three transforms, cosine, sine and Melline transforms, act continuously from L^2 to L^2 . Under these extra assumptions on the functions $x(t), \hat{x}_c(t), \hat{x}_s(t)$, the Melline transforms $\Phi_x(\zeta), \Phi_{\hat{x}_c}(\zeta), \Phi_{\hat{x}_s}(\zeta)$ are defined everywhere on the vertical line $\operatorname{Re} \zeta = \frac{1}{2}$ and are continuous functions there. For such x , the equalities (50) will be established for every $\zeta: \operatorname{Re} \zeta = \frac{1}{2}$.

We fix $\zeta: \operatorname{Re} \zeta = \frac{1}{2}$. The Melline transform $\Phi_{\hat{x}_c}(\zeta)$ is:

$$\Phi_{\hat{x}_c}(\zeta) = \lim_{R \rightarrow \infty} \int_0^R \hat{x}_c(t) t^{\zeta-1} dt.$$

Substituting the expression (49a) for $\hat{x}_c(t)$ into the last formula, we obtain:

$$\Phi_{\hat{x}_c}(\zeta) = \lim_{R \rightarrow \infty} \int_0^R \left(\sqrt{\frac{2}{\pi}} \int_0^\infty x(s) \cos(ts) ds \right) t^{\zeta-1} dt. \quad (51)$$

For fixed finite R , we change the order of integration:

$$\int_0^R \left(\int_0^\infty x(s) \cos(ts) ds \right) t^{\zeta-1} dt = \int_0^\infty x(s) \left(\int_0^R \cos(ts) t^{\zeta-1} dt \right) ds.$$

The change of order of integration is justified by Fubini's theorem. Changing the variable $ts = \tau$, we get

$$\int_0^R \cos(ts) t^{\zeta-1} dt = s^{-\zeta} \int_0^{Rs} \cos(\tau) \tau^{\zeta-1} d\tau.$$

Thus

$$\begin{aligned}
\int_0^R \left(\sqrt{\frac{2}{\pi}} \int_0^\infty x(s) \cos(ts) ds \right) t^{\zeta-1} dt = \\
= \int_0^\infty x(s) s^{-\zeta} \left(\sqrt{\frac{2}{\pi}} \int_0^{Rs} \cos(\tau) \tau^{\zeta-1} d\tau \right) ds. \quad (52)
\end{aligned}$$

According to Lemma 1, for every $s > 0$,

$$\lim_{R \rightarrow \infty} \int_0^{Rs} \cos(\tau) \tau^{\zeta-1} d\tau = \left(\cos \frac{\pi}{2} \zeta \right) \Gamma(\zeta),$$

The value $\int_0^\rho (\cos \tau) \tau^{\zeta-1} d\tau$, considered as a function of ρ , vanishes at $\rho = 0$, is a continuous function of ρ , and has a finite limit as $\rho \rightarrow \infty$. Therefore there exist a finite $M(\zeta) < \infty$ such that the estimate holds: $\left| \int_0^\rho (\cos \tau) \tau^{\zeta-1} d\tau \right| \leq M(\zeta)$, where the value $M(\zeta)$ does not depend on ρ . In other words,

$$\left| \int_0^{Rs} \cos(\tau) \tau^{\zeta-1} d\tau \right| \leq M(\zeta) < \infty \quad \forall s, R : 0 \leq s < \infty, 0 \leq R < \infty.$$

By the Lebesgue theorem on dominating convergence,

$$\begin{aligned}
\lim_{R \rightarrow \infty} \int_0^\infty x(s) s^{-\zeta} \left(\sqrt{\frac{2}{\pi}} \int_0^{Rs} \cos(\tau) \tau^{\zeta-1} d\tau \right) ds = \\
= \int_0^\infty x(s) s^{-\zeta} \left(\sqrt{\frac{2}{\pi}} \int_0^\infty \cos(\tau) \tau^{\zeta-1} d\tau \right) ds. \quad (53)
\end{aligned}$$

Taking into account the equalities (51), (53) and using (16a) and (20a), we reduce the last equality to the form

$$\Phi_{\hat{x}_c}(\zeta) = \int_0^\infty x(s) s^{-\zeta} ds \cdot 2^{\zeta-\frac{1}{2}} \frac{\Gamma(\frac{\zeta}{2})}{\Gamma(\frac{1}{2}-\frac{\zeta}{2})}.$$

To obtain (50a) from the previous equality, we need only consider that

$$\int_0^{\infty} x(s) s^{-\zeta} ds = \Phi_x(1 - \zeta).$$

The equality (50b) can be proved analogously. \square

Remark 8. *The equalities (50) can be presented in the form*

$$\Phi_{\hat{x}_c}\left(\frac{1}{2} + i\tau\right) = \Phi_x\left(\frac{1}{2} - i\tau\right) \cdot c^2(\tau), \quad (54a)$$

$$\Phi_{\hat{x}_s}\left(\frac{1}{2} + i\tau\right) = \Phi_x\left(\frac{1}{2} - i\tau\right) \cdot s^2(\tau), \quad (54b)$$

where $c(\tau)$ and $s(\tau)$ were introduced in (30).

Proof of Theorem 3. Let $x_c(t)$ be defined by (49a). The equality $\mathbf{C}x = x$, i.e. the equality $x_c(t) = x(t)$ for functions $x_c(t)$, $x(t)$, is equivalent to the equality

$$\Phi_{\hat{x}_c}\left(\frac{1}{2} + i\tau\right) = \Phi_x\left(\frac{1}{2} + i\tau\right)$$

for their Melline transforms. According to Lemma 6, (54a), the last equality can be reduced² to the form

$$\Phi_x\left(\frac{1}{2} - i\tau\right) \cdot c(\tau) = \Phi_x\left(\frac{1}{2} + i\tau\right) \cdot c(-\tau), \quad -\infty < \tau < \infty. \quad (55a)$$

Analogously, the equalities $\mathbf{C}x = -x$, $\mathbf{S}x = x$ and $\mathbf{S}x = -x$ for the functions $x(t)$ are equivalent to the equalities

$$\Phi_x\left(\frac{1}{2} - i\tau\right) \cdot c(\tau) = -\Phi_x\left(\frac{1}{2} + i\tau\right) \cdot c(-\tau), \quad -\infty < \tau < \infty, \quad (55b)$$

and

$$\Phi_x\left(\frac{1}{2} - i\tau\right) \cdot s(\tau) = \Phi_x\left(\frac{1}{2} + i\tau\right) \cdot s(-\tau), \quad -\infty < \tau < \infty. \quad (56a)$$

$$\Phi_x\left(\frac{1}{2} - i\tau\right) \cdot s(\tau) = -\Phi_x\left(\frac{1}{2} + i\tau\right) \cdot s(-\tau), \quad -\infty < \tau < \infty, \quad (56b)$$

Thus each of the equalities $\mathbf{C}x = x$, $\mathbf{C}x = -x$, $\mathbf{S}x = x$, $\mathbf{S}x = -x$ for the function $x(t)$, $0 < t < \infty$, is equivalent to the symmetry condition for its

² Remember that $c^{-1}(\tau) = c(-\tau)$.

Melline transform $\Phi_x\left(\frac{1}{2} + i\tau\right)$, $-\infty < \tau < \infty$. These symmetry conditions, which appear as conditions (55), (56), can be presented in the form

$$\begin{aligned}\Phi_x\left(\frac{1}{2} + i\tau\right) &= \sqrt{\pi} c(\tau) \phi(|\tau|), & -\infty < \tau < \infty, \\ \Phi_x\left(\frac{1}{2} + i\tau\right) &= \frac{1}{i} \operatorname{sign}(\tau) \sqrt{\pi} c(\tau) \phi(|\tau|), & -\infty < \tau < \infty,\end{aligned}$$

and

$$\begin{aligned}\Phi_x\left(\frac{1}{2} + i\tau\right) &= \sqrt{\pi} s(\tau) \phi(|\tau|), & -\infty < \tau < \infty, \\ \Phi_x\left(\frac{1}{2} + i\tau\right) &= \frac{1}{i} \operatorname{sign}(\tau) \sqrt{\pi} s(\tau) \phi(|\tau|), & -\infty < \tau < \infty,\end{aligned}$$

where $\phi(\tau)$ is function defined for $0 < \tau < \infty$. Comparing these expressions for the function $\Phi_x\left(\frac{1}{2} + i\tau\right)$ with the expressions (28), (29) for the eigenfunctions $e_c^+(t, \tau)$, $e_c^-(t, \tau)$, $e_s^+(t, \tau)$, $e_s^-(t, \tau)$, we see that in each of the four cases, the inversion formula

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} t^{-\frac{1}{2} + i\tau} \Phi_x\left(\frac{1}{2} + i\tau\right) d\tau$$

for the Melline transform can be presented in terms of the function $\phi(\tau)$ as

$$x(t) = \int_0^{\infty} e_c^+(t, \tau) \phi(\tau) d\tau, \quad x(t) = \int_0^{\infty} e_c^-(t, \tau) \phi(\tau) d\tau, \quad (57a)$$

$$x(t) = \int_0^{\infty} e_s^+(t, \tau) \phi(\tau) d\tau, \quad x(t) = \int_0^{\infty} e_s^-(t, \tau) \phi(\tau) d\tau, \quad (57b)$$

respectively. Now the symmetries (55), (56) of the function $\Phi_x\left(\frac{1}{2} + i\tau\right)$ are hidden in the structure of functions e_c^+ , e_c^- , e_s^+ , e_s^- .

Thus, the equalities $\mathbf{C}x = x$, $\mathbf{C}x = -x$ and $\mathbf{S}x = x$, $\mathbf{S}x = -x$ for the functions x are equivalent to representability of x in one of the four forms (57), i.e. in the form $x = \mathbf{T}_c^+ \phi$, $x = \mathbf{T}_c^- \phi$, $x = \mathbf{T}_s^+ \phi$ and $x = \mathbf{T}_s^- \phi$, respectively (with $\phi \in L^2(\mathbb{R}_+)$). \square

Acknowledgements I thank Armin Rahn for his careful reading of the manuscript and his help in improving the English in this paper.

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